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The need for such a scheme was suggested by Professor D. H. Lehmer of the University of California at Berkeley.

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Chebyshev Approximations to the Gamma Function

By Helmut Werner and Robert Collinge

In this note several Chebyshev approximations are given for the function $y = \Gamma(x + 2)$ for x in the $0 \leq x \leq 1.0$ range. The approximations were obtained from a table of $\Gamma(x + 2)$, employing well-known methods as described in numerous papers; see for instance [1] and the literature quoted there. The table of $\Gamma(x + 2)$ was calculated from the asymptotic expansion of $\log \Gamma(z)$ as given in [2] to provide data accurate to at least 10^{-21} . Compare also [3].

The asymptotic expansion of $\ln \Gamma(z)$ is given by

$$\ln \Gamma(z) = (z - \frac{1}{2}) \ln z - z + \ln \sqrt{2\pi} + \Phi(z)$$

where

$$\Phi(z) = \sum_{r=1}^n \frac{(-1)^{r-1} B_r}{2r(2r-1)} \frac{1}{z^{2r-1}} + R_n(z),$$

and B_r is the r th Bernoulli number.

It can be shown [2] that for $z > 0$ the value of $\Phi(z)$ always lies between the sum of n terms and the sum of $(n + 1)$ terms of the series, for all values of n . In terminating this series with the n th term the error $R_n(z)$ will be less than

$$\frac{B_{n+1}}{2(n+1)(2n+1)} \cdot \frac{1}{z^{2n+1}}.$$

By truncating $\Phi(z)$ at the 10th term it is easily shown that for values of $z \geq 13$, the error in the expansion is less than 5.5×10^{-22} . We therefore replace $\Phi(z)$ by $\sum_{i=1}^{10} A_i/z^{2i-1}$ and calculate $\ln \Gamma(z)$ for values of z in the range $13 \leq z \leq 14$.

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TABLE 1
Table of Coefficients

n	7	8	10
$\epsilon_{\max}^{(n)}$	0.25×10^{-1}	0.16×10^{-2}	0.74×10^{-11}
$\nu = 0$	0.99999 99758	0.99999 99998 452	0.99999 99999 9269
1	0.42278 74605	0.42278 43662 730	0.42278 43369 6202
2	0.41177 41955	0.41183 92935 920	0.41184 02517 9616
3	0.08211 17404	0.08159 03449 474	0.08157 82187 8492
4	0.07211 01567	0.07416 00915 535	0.07423 79076 0629
5	0.00445 11400	0.00007 55964 181	-0.00021 09074 6731
6	0.00515 89951	0.01033 20685 065	0.01097 36958 4174
7	0.00160 63118	-0.00157 80074 635	-0.00246 67479 8051
8		0.00079 62464 760	0.00153 97681 0472
9			-0.00034 42342 0456
10			0.00006 77105 7117

n	13	15
$\epsilon_{\max}^{(n)}$	0.96×10^{-14}	0.97×10^{-16}
$\nu = 0$	0.99999 99999 99990 44	0.99999 99999 99999 9032
1	0.42278 43351 02334 79	0.42278 43350 98518 1178
2	0.41184 03301 66781 29	0.41184 03304 21981 4831
3	0.08157 69261 24155 46	0.08157 69194 01388 6786
4	0.07424 89154 19444 74	0.07424 90079 43401 2692
5	-0.00026 61865 94953 06	-0.00026 69510 28755 5266
6	0.01114 97143 35778 93	0.01115 38196 71906 6992
7	-0.00283 64625 20372 82	-0.00285 15012 43034 6494
8	0.00206 10918 50225 54	0.00209 97590 35077 0629
9	-0.00083 75646 85135 17	-0.00090 83465 57420 0521
10	0.00037 53650 52263 07	0.00046 77678 11496 4956
11	-0.00012 14173 48706 32	-0.00020 64476 31915 9326
12	0.00002 79832 88995 83	0.00008 15530 49806 6373
13	-0.00000 30301 90810 28	-0.00002 48410 05384 8712
14		0.00000 51063 59207 2582
15		-0.00000 05113 26272 6698

n	17	18
$\epsilon_{\max}^{(n)}$	0.10×10^{-17}	0.10×10^{-18}
$\nu = 0$	0.99999 99999 99999 99901 2	0.99999 99999 99999 99990 02
1	0.42278 43350 98467 79580 6	0.42278 43350 98467 21319 61
2	0.41184 03304 26367 20638 1	0.41184 03304 26430 62304 23
3	0.08157 69192 50260 90503 9	0.08157 69192 47528 84581 87
4	0.07424 90106 80990 41696 9	0.07424 90107 42091 91715 38
5	-0.00026 69810 33348 38176 8	-0.00026 69818 88740 38315 07
6	0.01115 40360 24034 39169 2	0.01115 40438 29069 91793 23
7	-0.00285 25821 44619 65607 6	-0.00285 26318 64792 11862 89
8	0.00210 36287 02459 83329 2	0.00210 38579 20672 20524 09
9	-0.00091 84843 69099 08014 2	-0.00091 92675 95039 95026 11
10	0.00048 74227 94476 75810 4	0.00048 94361 06298 14458 31
11	-0.00023 47204 01891 94985 9	-0.00023 86428 33752 63647 10
12	0.00011 15339 51966 59947 0	0.00011 73283 10224 09396 51
13	-0.00004 78747 98383 44672 4	-0.00005 43183 86280 13508 99
14	0.00001 75102 72717 90508 0	0.00002 28140 41153 66022 75
15	-0.00000 49203 75090 42313 2	-0.00000 80523 43363 48309 16
16	0.00000 09199 15340 71621 4	0.00000 21741 77495 45332 61
17	-0.00000 00839 94049 59039 7	-0.00000 03889 70057 33769 55
18		0.00000 00339 81801 01810 43

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2. E. T. V
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3. M. E. S
v. 13, 1959, p.
4. H. S. U
1938, p. 23-39
5. H. S. U
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For the convenience of the reader the A_i coefficients are quoted below, to 25 significant figures.

$$\begin{aligned} A_1 &= 0.08333 \ 33333 \ 33333 \ 33333 \ 33333 \ 3 \\ A_2 &= -0.00277 \ 77777 \ 77777 \ 77777 \ 77777 \ 78 \\ A_3 &= 0.00079 \ 36507 \ 93650 \ 79365 \ 07936 \ 508 \\ A_4 &= -0.00059 \ 52380 \ 95238 \ 09523 \ 80952 \ 381 \\ A_5 &= 0.00084 \ 17508 \ 41750 \ 84175 \ 08417 \ 508 \\ A_6 &= -0.00191 \ 75269 \ 17526 \ 91752 \ 69175 \ 27 \\ A_7 &= 0.00641 \ 02564 \ 10256 \ 41025 \ 64102 \ 56 \\ A_8 &= -0.02955 \ 06535 \ 94771 \ 24183 \ 00653 \ 6 \\ A_9 &= 0.17964 \ 43723 \ 68830 \ 57316 \ 49385 \\ A_{10} &= -1.39243 \ 22169 \ 05901 \ 11642 \ 7432 \\ \ln \sqrt{2\pi} &= 0.91893 \ 85332 \ 04672 \ 74178 \ 03297 \end{aligned}$$

A triple precision logarithm routine was used to evaluate $\ln z$, and then an exponential routine to calculate $\Gamma(z) = e^{\ln \Gamma(z)}$. Each of these routines produces results accurate to at least 24 significant digits.

After obtaining a table of $\Gamma(z)$ for z in the range $13 \leq z \leq 14$, we made use of the recursion formula $\Gamma(z+1) = z\Gamma(z)$ in order to obtain a table of $\Gamma(x+2)$ for x in the range $0 \leq x \leq 1.0$.

From the tests made on the results obtained, the values of $\Gamma(x+2)$ were shown to be accurate to at least 21 significant figures.

Several Chebyshev approximations have been calculated to provide varying degrees of accuracy. Let

$$\Gamma(2+x) = \sum_{r=0}^n a_r^{(n)} x^r + \epsilon_n(x)$$

and

$$\epsilon_{\max}^{(n)} = \max_{0 \leq x \leq 1} |\epsilon_n(x)|.$$

Table 1 gives the coefficients $a_r^{(n)}$ for $n = 7, 8, 10, 13, 15, 17, 18$ together with the corresponding $\epsilon_{\max}^{(n)}$.

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